INTERPOLATION BY SPLINES SATISFYING MIXED BOUNDARY CONDITIONS

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ABSTRACT

We consider interpolation of Hermite data by splines of degree n with k given knots, satisfying boundary conditions which may involve derivatives at both end points (e.g., a periodicity condition). It is shown that, for a certain class of boundary conditions, a necessary and sufficient condition for the existence of a unique solution is that the data points and knots interlace properly and that there does not exist a polynomial solution of degree $n-k$. The method of proof is to show that any spline interpolating zero data vanishes identically, rather than the usual determinantal approach.

1. Introduction

Proving the existence and uniqueness of a polynomial interpolating Hermite data may be accomplished either by proving that the Vandermonde determinant is non-zero, or by showing that any polynomial interpolating zero data vanishes identically. In analyzing spline interpolation, the determinantal approach has been used extensively, starting with Schoenberg and Whitney [10] and culminating with Karlin [5]. In contrast, the second approach has been largely neglected because it involves the use of Rolle's theorem or, when boundary conditions are imposed, of the Budan-Fourier theorem, and no spline versions of these theorems were available. However, having developed the Budan-Fourier theorem for splines in a previous paper [7], we are now in a position to consider interpolation by splines via the second approach. Moreover, on doing so, there becomes apparent a natural way to extend Karlin's results [5] on interpolation by splines satisfying separated boundary conditions so as to include also mixed boundary conditions, of which periodic

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ones are an example (in this context see Karlin & Lee [4]). In order to describe this extension, our main new result, let us formulate the interpolation problem.

Consider interpolation of data $f(x_i)$ at the ordered set of points $X = \{x_i\}_1^{n+k+1-r}$ in (a,b) by a polynomial spline $S(x)$ of degree n which has prescribed knots $\Xi = \{\xi_i\}_1^k$ and satisfies the boundary conditions

$$\sum_{j=0}^{n} [a_{ij}S^{(j)}(a) + b_{ij}S^{(n-j)}(b)] = u_i, \quad i = 1, \cdots, r. \tag{1}$$

Note that the coefficients b_{ij} are not in the usual order.

The following stipulations concerning the knots and data points will be assumed to prevail throughout the sequel.

1) R consecutive ξ's may coincide, $R \leqq n+1$ with the interpretation that at such a point $S(x)$ has a knot of multiplicity R, i.e., $S^{(i)}(x)$, $i = 0, \cdots, n-R$, is continuous at ξ.

2) Coincidences of x_i's in the set X are permitted as follows:

a) at a point $\bar{x}$ in (a,b) different from a knot M successive x_i, $M \leqq n+1$, are allowed to coincide with the interpretation of Hermite interpolation at that point:

$$S^{(i)}(\bar{x}) = f^{(i)}(\bar{x}), \qquad i = 0, \cdots, M-1.$$

b) at a knot ξ of multiplicity R left and right Hermite interpolation is permitted with $M^- \leqq n+1$, $M^+ \leqq n+1$, i.e.,

$$S^{(i)}(\xi^-) = f^{(i)}(\xi^-), \qquad i = 0, \cdots, M^- - 1 \quad \text{and}$$

$$S^{(i)}(\xi^+) = f^{(i)}(\xi^+), \qquad i = 0, \cdots, M^+ - 1.$$

If $M^+ = M^- = M \leqq n+1-R$, then the point ξ is included M times in the set X; otherwise, the set includes $M^- - (n+1-R)$ times the point $\xi^-, n+1-R$ times the point ξ and $M^+ - (n+1-R)$ times the point ξ^+ in precisely that order.

As for the boundary form, given k, the total number of knots counting multiplicities, form the $r \times m$ matrices:

$$\tilde{A}_{r,m} = \|(-1)^{n-k-j}a_{ij}\|_{i=1,j=0}^{r\ \ m-1} \text{ and } B_{r,m} = \|b_{ij}\|_{i=1,j=n-m+1}^{r\ \ n}, \tag{2}$$

and the compound matrix $\|\tilde{A}_{r,m}, B_{r,m}\|$. We will denote rank $\tilde{A}_{r,n+1} = p$ and rank $B_{r,n+1} = q$. The following is the basic requirement on the boundary form.

Postulate I. $\|\tilde{A}_{r,n+1}, B_{r,n+1}\|$ *is sign consistent of order* r (SC_r) *and has rank* r (a matrix U is said to be SC_r if all $r \times r$ non-zero subdeterminants of U have the same sign).

The importance of the sign consistency condition and its connection with the Budan-Fourier theorem was brought to the fore in Karlin [5] and Karlin and Micchelli [6] (see also Karlin [3, ch. 10]) and will be explored further in Section 2.

As usual with splines, unique interpolation will be impossible unless the points fulfill an interlacing condition which here takes on the following form.

INTERLACING CONDITION — FORMULATION 1. *Given the boundary conditions* (1), *it will be said that the sets of points* $\{x_i\}_1^{n+k+1-r}$ *and knots* $\{\xi_i\}_1^k$ *fulfill the interlacing condition if there exists an integer* λ, $r-q \leqq \lambda \leqq p$, *such that*

$$x_{i-\lambda} < \xi_i < x_{i+n+1-\lambda} \qquad i = 1,\cdots,k \tag{3}$$

wherever it makes sense; except that equalities are allowed

i) *at the left hand if* $x_{i-\lambda} = \xi_i^-$

ii) *at the right hand if* $x_{i+n+1-\lambda} = \xi_i^+$.

An alternative formulation of this condition will be presented in Section 2. We are now ready to state the interpolation theorem.

THEOREM 1. *Let there be given the set of knots* $\{\xi_i\}_1^k$, *data at the points* $\{x_i\}_1^{n+k+1-r}$ *and the boundary conditions* (1). *Then the interlacing condition is a necessary condition for the existence of a unique spline of degree n with the knots* Ξ *which interpolates to the data and satisfies the boundary conditions.*

If $k \geqq \min(n, r-p, r-q)$ *then this condition is also sufficient to ensure a unique solution.*

If $k < \min(n, r-p, r-q)$ *and the interlacing condition holds then there is a unique solution if and only if*

$$n+k+1-r+\operatorname{rank}\|\tilde{A}_{r,m}, B_{r,m}\| \geqq m \quad \textit{for} \quad 1 \leqq m \leqq n+1-k.$$

The main body of the proof of this theorem is contained in Section 3, whereas in Section 2 we develop the tools for it, examining in particular the connection between the Budan-Fourier theorem and the boundary conditions. Finally in Section 4, a few examples are presented.

2. Preliminaries. The Budan-Fourier theorem and its connection with the boundary conditions, the interlacing condition

Consider a spline $t(x)$ of degree n defined in (a,b) with the knots η_i, $i = 1,\cdots,L$, of respective multiplicities R_i. In between knots η_i, η_{i+1}, $i = 0,\cdots,k$, $t(x)$ is a polynomial whose degree we denote by n_i. We will assume that $t(x)$ does not vanish identically anywhere in (a, b) and that it is of degree n exactly, i.e., $\max_{0 \leq i \leq k} n_i = n$.

We define the multiplicity, $Z(t; \bar{x})$, of a zero of $t(x)$ at $\bar{x}$ by the usual polynomial definition if $\bar{x}$ is different from a knot. If $\bar{x}$ coincides with a knot η of multiplicity R, at which $t(x)$ has a left hand zero of multiplicity α and a right hand zero of multiplicity β, then $Z(t; \eta)$ is defined as:

i) α if $\alpha = \beta \leqq n - R$;

ii) $\alpha + S^+(t^{(\alpha)}(\eta^-), t^{(n+1-R)}(\eta^+))$, if $\alpha \geqq \beta = n + 1 - R$;

iii) $\beta + S^+(t^{(n+1-R)}(\eta^-), (-1)^{n+1-R-\beta} t^{(\beta)}(\eta^+))$, if $\beta \geqq \alpha = n + 1 - R$; (4)

where $S^+(c_i)_0^m \equiv S^+(\{c_i\}_0^m) \equiv S^+(c_0, c_1, \cdots, c_m)$ equals the maximum number of sign changes in the ordered sequence $c_0, \cdots, c_m$ when each zero is allowed to be $+1$ or -1.

Denote by $Z(t; (a, b))$ the total number of zeros of $t(x)$ in (a, b), counting multiplicities. Then from [7], we obtain the following bound based on the Budan-Fourier theorem for splines.

Lemma 1. *Let $t(x)$ be of degree n exactly, $t(x) \not\equiv 0$ everywhere in (a, b). Let μ, ν be the largest integers such that $t^{(\mu)}(a) \neq 0$, $t^{(\nu)}(b) \neq 0$. Then*

$$Z(t; (a, b)) + S^+((-1)^i t^{(i)}(a))_0^n + S^+(t^{(i)}(b))_0^n \tag{5}$$
$$\leqq n + \sum_{i=1}^{L} R_i - S^+(t^{(\mu)}(a), (-1)^{n-\nu} \varepsilon t^{(\nu)}(b))$$

where $\varepsilon = (-1)^{\sum_{i=1}^{L} R_i}$. Moreover the right hand side equals $n + \sum_{i=1}^{L} R_i$ only if $n_j = n$, $j = 1, \cdots, L-1$ and

$$S^+(\{(-1)^{n-R_j+i} t^{(i)}(\eta_j^-)\}_{n_{j-1}}^{n-R_j}, \{t^{(i)}(\eta_j^+)\}_{n-R_j+1}^{n_j}) \tag{6}$$
$$= |n_{j-1} + R_j - n| + |n_j + R_j - n| \qquad j = 1, \cdots, L$$

i.e. each expression attains its maximum value.

We proceed to show how this result ties up with Postulate 1 concerning the boundary conditions. As stated in the introduction, we will prove the existence of an interpolating spline by showing that any spline $t(x)$ interpolating to zero data and satisfying the homogeneous boundary conditions has to vanish identically. A first step will be to show by contradiction, using Lemma 1, that $t(x)$ cannot be of exact degree n. On assuming the contrary, it follows that $\sum_{i=1}^{L} R_i = k$ and $Z(t; (a, b)) \geqq n + k + 1 - r$. Hence, (5) will lead to a contradiction if:

$$S^{+}((-1)^{i}t^{(i)}(a))_0^n + S^{+}(t^{(i)}(b))_0^n + S^{+}(t^{(\mu)}(a), (-1)^{n+k-\nu}t^{(\nu)}(b)) \geqq r, \qquad (7)$$

whenever $t(x)$ satisfies the homogeneous boundary conditions.

On writing $d_i = (-1)^{k+n-i}t^{(i)}(a)$, $i = 0, \cdots, n$, and $d_{n+1+i} = t^{(n-i)}(b)$, $i = 0, \cdots, n$, $d = (d_0, d_1, \cdots, d_{2n+1})$ and $U = \|\tilde{A}_{r,n+1}, B_{r,n+1}\|$, the above requirement becomes $S^{+}(d_i)_0^{2n+1} \geqq r$ whenever $Ud = 0$. We are thus led to the adoption of Postulate 1, that U is SC_r, on the basis of the following result whose proof can be found in Gantmacher and Krein [2] or Karlin [3, ch. 5].

Lemma 2. *Let the $r \times 2(n+1)$ matrix U, $r < 2(n+1)$, be of rank r. Then U is SC_r if, and only if, $S^{+}(d_i)_0^{2n+1} \geqq r$ for every vector d such that $Ud = 0$.*

We will also need the following extension of this result.

Lemma 3. *Let the $r \times 2(n+1)$ matrix $\|\tilde{A}_{r,n+1}, B_{r,n+1}\|$ be of rank r, $r \leqq 2(n+1)$, and SC_r. If $\|\tilde{A}_{r,l}, B_{r,m}\|$, $1 \leqq l \leqq n+1$, $1 \leqq m \leqq n+1$, is of rank ρ then either* (i) *$\rho = l + m$ or* (ii) *$S^{+}(y_i)_1^{l+m} \geqq \rho$ for every vector $y = (y_1, \cdots, y_{l+m})$ satisfying $\|(-1)^{r-\rho}\tilde{A}_{r,l}, B_{r,m}\|y = 0$.*

Proof. Note: $\tilde{A}_{r,l}$ and $B_{r,m}$ are as defined in (2). Assume for simplicity that the matrix $\|\tilde{A}_{r,n+1}, B_{r,n+1}\|$ is arranged in such a way that $a_{ij} = 0$, $1 \leqq i \leqq r-\rho$, $0 \leqq j \leqq l-1$ and $b_{ij} = 0$, $1 \leqq i \leqq r-\rho$, $n-m+1 \leqq j \leqq n$. Let U_1 be a non-zero $\rho \times \rho$ subdeterminant of $\|\tilde{A}_{r,l}, B_{r,m}\|$ with γ columns chosen from $\tilde{A}_{r,l}$, and U_2 a fixed non-zero $(r-\rho)\times(r-\rho)$ subdeterminant of $\| \; \|(-1)^{k+n-j}a_{ij}\|_{i=1,j=l}^{r-\rho\; n}$, $\|b_{ij}\|_{i=1,j=0}^{r-\rho\; n-m}\|$. Since $\|\tilde{A}_{r,n+1}, B_{r,n+1}\|$ is SC_r all its $r \times r$ non-zero subdeterminants have the same sign and in particular $(-1)^{\gamma(r-\rho)}U_1U_2$ has the same sign for any choice of U_1. Thus $\|(-1)^{r-\rho}\tilde{A}_{r,l}, B_{r,m}\|$ is SC_ρ and the lemma follows.

A final topic we want to touch upon before tackling the proof of the interpolation theorem is the interpretation of the interlacing condition. Since the interpolation and boundary conditions form a linear system determining the unknown coefficients of the spline it is natural to require that, restricting attention to any subinterval of (a, b), there shall be no more conditions on the spline than there are available parameters there. When doing this one has to take into account the fact that of the r boundary conditions $r-q$ apply only at the endpoint a (not involving b) and $r-p$ only at b (since rank $\tilde{A}_{r,n+1} = p$, rank $B_{r,n+1} = q$). Denote by $K(\eta_1, \eta_2)$ the number of knots in the interior of the interval (η_1, η_2) and by $N[\eta_1, \eta_2]$ the number of data points in the interval $[\eta_1, \eta_2]$ including any possible data points $\eta_1, \eta_1^+, \eta_2^-$ and η_2, but excluding η_1^- and η_2^+. Then the previous reasoning suggests the following formulation of the interlacing condition, compare Rice [9].

INTERLACING CONDITION — FORMULATION 2. *Given integers* p, q, r *such that* $p \leqq n+1$, $q \leqq n+1$, $p+q \geqq r \geqq \max(p,q)$ *the following set of conditions on the knots* $\{\xi_i\}_1^k$ *and data points* $\{x_i\}_1^{n+k+1-r}$ *is equivalent to the interlacing condition*:

1) $N(a,\xi_i]+r-q \leqq n+1+K(a,\xi_i)$ *for all* $1 \leqq i \leqq k$.
2) $N[\xi_i,b)+r-p \leqq n+1+K(\xi_i,b)$ *for all* $1 \leqq i \leqq k$.
3) $N[\xi_i,\xi_j] \leqq n+1+K(\xi_i,\xi_j)$ *for all* $\xi_i < \xi_j$.
4) $N(a,\xi_i]+N[\xi_j,b)+r \leqq 2(n+1)+K(a,\xi_i)+K(\xi_j,b)$ *for all* $\xi_i < \xi_j$.

PROOF. That Formulation 1 implies Formulation 2 is easily verified. To prove the converse, we begin by noting that 1 and 2 imply that $x_{i-p} < \xi_i < x_{i+n+1-r+q}$, $i=1,\cdots,k$ wherever it makes sense, with the exception that equality is permitted at the left if $x_{i-p} = \xi_i^-$, and at the right if $x_{i+n+1-r+q} = \xi_i^+$. Indeed, suppose to the contrary that e.g., $\xi_i < x_{i-p}$. Then $N[\xi_i,b)$ includes x_{i-p} hence $N[\xi_i,b) \geqq n+k+1-r-i+p+1 \geqq n+2+p-r+K(\xi_i,b)$ because $K(\xi_i,b) \leqq k-i$. This contradicts 2.

We conclude therefore that either $x_{n+k+1-r} < \xi_1$ in which case we can simple take $\lambda = r-q$, or else there is a least integer l, $l \leqq p$, such that $x_{i-l} < \xi_i$ or $x_{i-l} = \xi_i^-$, $i=1,\cdots,k$. Since l is least, there must be an index i_1, for which

$$\xi_{i_1} \leqq x_{i_1+1-l} \quad \text{or} \quad x_{i_1+1-l} = \xi_{i_1}^+ .$$

Suppose now, contrary to Formulation 1, that there exists an index i_2 for which $\xi_{i_2} > x_{i_2+n+1-l}$. Necessarily $\xi_{i_1} \neq \xi_{i_2}$ because $\xi_{i_1} = \xi_{i_2}$ entails $x_{i_2+n+1-l} < x_{i_1+1-l}$ or $i_1 - i_2 \geqq n+1$, an impossibility since at most $n+1$ ξ's coincide. If $i_1 < i_2$, at the very least the data points $x_{i_1+1-l},\cdots,x_{i_2+n+1-l}$ are included in $N[\xi_{i_1},\xi_{i_2}]$. Thus $N[\xi_{i_1},\xi_{i_2}] \geqq n+2+K(\xi_{i_1},\xi_{i_2})$, contradicting 3. Similarly, if $i_1 > i_2$, a contradiction with 4 ensues.

Having thus established the existence of one integer, l, for which $x_{i-l} < \xi_i < x_{i+n+1-l}$ with exceptions as noted, we infer the existence of a largest integer $\lambda, \lambda \leqq p$ for which this is true. Necessarily for this λ, it holds that $\lambda \geqq r-q$, for were this not the case, it would imply the existence of an index j for which, say, $\xi_j > x_{j+n+1-(\lambda+1)} \geqq x_{j+n+1-(r-q)}$, which we saw cannot happen.

REMARK. When the exceptions (i) and (ii) of Formulation 1 occur, necessarily $N(a,\xi_i] = n+1-\lambda+K(a,\xi_i)$, and $N[\xi_i,b) = n+1-r+\lambda+K(\xi_i,b)$. For while in general these are only upper bounds, when e.g., $x_{i-\lambda} = \xi_i^-$, ξ_i being a knot of multiplicity R, it is implied that $N(a,\xi_i]$ and $N[\xi_i,b)$ have exactly $n+1-R$ data points in common. Hence

$N(a,\xi_i]+N[\xi_i,b) = n+k+1-r+n+1-R = 2(n+1)-r+K(a,\xi_i)+K(\xi_i,b)$.

3. Proof of the interpolation theorem

The interpolation and boundary conditions yield a system of linear equations for the determination of the interpolating spline coefficients (expanded in any basis). The conditions under which there exists a unique solution are therefore the same conditions which ensure that any spline $t(x)$ of degree n satisfying

$$
\begin{aligned}
&t(x_i)=0 && i=1,\cdots,n+k+1-r \\
&\sum_{j=0}^{n}\left[a_{ij}t^{(j)}(a)+b_{ij}t^{(n-j)}(b)\right]=0 && i=1,\cdots,r
\end{aligned}
\tag{8}
$$

vanishes identically. This is the approach we base our proof on. For ease of reading we present the main arguments as separate lemmas.

LEMMA 4. *The interlacing condition is necessary for the existence of a unique solution.*

PROOF. It will be convenient to use Formulation 2 of the interlacing condition. We will show that when one of its conditions does not hold then it is possible to construct a non-trivial spline satisfying (8).

Assume for example that $N[\eta,b)+r-p>n+1+K(\eta,b)$ with η a knot of multiplicity R. Let (a,η) contain the interior knots $\xi_1,\cdots,\xi_l$ and the data points $x_1,\cdots,x_m$ excluding possible points η,η^+. Then $l=k-R-K(\eta,b)$, $m=n+k+1-r-N[\eta,b)$ and hence $m+n+1-R+p<n+1+l$. It is therefore possible to find a non-trivial spline $t_1(x)$ of degree n, defined on (a,η) with the knots $\xi_1,\cdots,\xi_l$ such that

$$t_1(x_i)=0,\quad i=1,\cdots,m \quad\text{and}\quad t_1^{(i)}(\eta)=0,\quad i=0,\cdots,n-R$$

$$\sum_{j=0}^{n}a_{ij}t_1^{(j)}(a)=0,\qquad i=1,\cdots,r.$$

Note that there are only p independent equations among the last r. This spline can be extended to a spline satisfying all the conditions by defining $t(x)=t_1(x)$ for $x\leqq\eta$ and $t(x)=0$ for $x\geqq\eta$.

Similarly if $N[\eta_1,\eta_2]>n+1+K(\eta_1,\eta_2)$, where η_1,η_2 are knots of respective multiplicities R_1,R_2, it means there are too many interpolation conditions in $[\eta_1,\eta_2]$ which can then be satisfied simply by letting $t(x)\equiv0$ in (η_1,η_2), while finding a non-trivial solution in (a,η_1) and (η_2,b). For the latter involves solving a system of $n+k+1-N[\eta_1,\eta_2]+n+1-R_1+n+1-R_2$ linear equations, yielded by the boundary conditions, the interpolation conditions in (a,η_1) and

(η_2, b) and also $t^{(i)}(\eta_1) = 0$, $i = 0, \cdots, n - R_1$, $t^{(i)}(\eta_2) = 0$, $i = 0, \cdots, n - R_2$, for the $2(n+1) + k - K(\eta_1, \eta_2) - R_1 - R_2$ coefficients which determine the spline in (a, η_1) and (η_2, b).

The remaining conditions can be established by analogous considerations which are therefore omitted.

LEMMA 5. *Let the interlacing condition hold. If $k \leqq n$ then $t(x)$ is a polynomial of degree at most $n - k$, and if $k > n$ then $t(x) \equiv 0$.*

PROOF. Note that $t^{(n-k)}(x)$ is always continuous since the worst discontinuity that $t(x)$ can have occurs when all the k knots ξ_i coincide, causing a discontinuity only in the derivative of order $n + 1 - k$.

Assume first of all that $t(x) \not\equiv 0$ everywhere in (a, b), and that at any knot there is only one-sided Hermite interpolation. Then all the points x_i are actual zeros of $t(x)$, hence

$$Z(t; (a, b)) \geqq n + k + 1 - r. \tag{9}$$

Suppose now by contradiction that $t(x)$ is a spline of degree m exactly, $n + 1 - k \leqq m \leqq n$. Distinguishing between three cases, we reach a contradiction by using the Budan-Fourier theorem and Lemma 2, in particular its consequence (7).

a) $m = n$. This is the case examined in Section 2. From Lemma 1:

$$Z(t; (a, b)) \leqq n + k - S^+((-1)^i t^{(i)}(a))_0^n - S^+(t^{(i)}(b))_0^n$$
$$- S^+(t^{(\mu)}(a), (-1)^{n+k-\nu} t^{(\nu)}(b)),$$

while from (7) the right hand does not exceed $n + k - r$, contradicting (9).

b) $n + 1 - k \leqq m \leqq n - 1$ and the number of distinct knots ξ_i is at least two. Since $t(x)$ is of degree m, instead of n, each knot which originally was of multiplicity R is now of multiplicity $\max(R - n + m, 0)$. There being at least two distinct knots, the total number of knots, counting multiplicities, is therefore at most $k - (n - m) - 1$, hence

$$Z(t; (a, b)) \leqq m + k - (n - m) - 1 - S^+((-1)^i t^{(i)}(a))_0^m - S^+(t^{(i)}(b))_0^m$$

and again a contradiction with (9) is reached on using (7) and

$$S^+(t^{(i)}(b))_0^n = S^+(t^{(i)}(b))_0^m + n - m.$$

c) $n + 1 - k \leqq m \leqq n - 1$ and all the knots ξ_i coincide. The multiplicity of the knot is thus $k - (n - m)$ and

$$Z(t;(a,b)) \leqq m+k-(n-m)-S^{+}((-1)^{i}t^{(i)}(a))_{0}^{m}-S^{+}(t^{(i)}(b))_{0}^{m}$$
$$-S^{+}(t^{(\mu)}(a),(-1)^{n+k-\nu}t^{(\nu)}(b))$$

which (7) shows to conflict with (9).

We consider now how these arguments are to be modified when it happens that $t(x)\equiv 0$ in some subinterval $(\eta_1,\eta_2), \eta_1,\eta_2$ knots of multiplicities $R_1, R_2, R_1 \leqq R_2$ say. Contracting this interval to a point, examine instead of $t(x)$ the spline $t_1(x)$ defined on $(a+\Delta, b)$, $\Delta=\eta_2-\eta_1$, by $t_1(x)=t(x-\Delta)$ for $a+\Delta \leqq x \leqq \eta_2$, $t_1(x)=t(x)$ for $\eta_2 \leqq x \leqq b$. This spline has the knots $\xi_i+\Delta$ when $\xi_i<\eta_1, \xi_i$ when $\xi_i>\eta_2$ as well as a knot of multiplicity R_2 at η_2; it vanishes at the points $x_i+\Delta$ when $x_i+\eta_1$, x_i when $x_i>\eta_2$ and also $t^{(i)}(\eta_1^{-})=0$, $i=0,\cdots,n-R_1$. Thus it has $k_1=k-K(\eta_1,\eta_2)-R_2$ knots and vanishes at

$$n+k+1-r-N[\eta_1,\eta_2]+n+1-R_2 \geqq n+k_1+1-r$$

points which may easily be seen to fulfill the interlacing conditions. Additionally, $t_1(x)$ satisfies the boundary conditions. However, the sign consistency condition (Postulate I) involves the number k which now does not equal the number of knots. Therefore, some caution is needed only in the case of truly mixed boundary conditions. Taking as an example the case $m=n$, and proceeding as in a, we do not necessarily arrive at a contradiction though it can be prevented only by assuming $Z(t;(a,b))=n+k_1+1-r$ and $S^{+}((-1)^{i}t^{(i)}(a))_0^n+S^{+}(t^{(i)}(b))_0^n=r-1$. Thus the equality condition of Lemma 1 is applicable and in particular

$$S^{+}(\{(-1)^{n-R_2}\,{}^{i}t_1^{(i)}(\eta_2^{-})\}_n^{n-R_2}, \{t_1^{(i)}(\eta_2^{+})\}_{n+1-R_2}^{n})=2R_2$$

implying $S^{+}(t_1^{(n+1-R_1)}(\eta_2^{-}), t_1^{(n+1-R_2)}(\eta_2^{+}))=1$. Hence, consulting definition (4), $Z(t_1;\eta_2)=n+2-R_1$ rather than $n+1-R_1$ and accordingly $Z(t_1;(a,b))=n+k_1+2-r$, the contradiction sought after. The case of two sided Hermite interpolation at a knot η of multiplicity R, with $M^{+}\geqq M^{-}>n+1-R$, can be handled similarly. For then $Z(t;\eta)=M^{+}$, instead of $M^{+}+M^{-}-(n+1-R)$ but also η becomes a knot of multiplicity $n+1-M^{-}$ instead of R. An equivalent interpolation problem is thus obtained except that the total number of knots no longer equals k. This completes the proof of Lemma 4.

We have shown therefore that the interlacing condition is also a sufficient condition for uniqueness when $k>n$. If however $k\leqq n$, then there may exist a polynomial of degree at most $n-k$ which satisfies all the conditions. An additional condition is therefore needed to ensure that the system of equations

$$t(x_i)=0, \qquad i=1,\cdots,n+k+1-r$$

$$\sum_{j=0}^{n-k}[a_{ij}t^{(j)}(a)+b_{ij}t^{(n-j)}(b)]=0 \qquad i=1,\cdots,r \tag{10}$$

has only the zero polynomial for a solution (is poised). Though the question has not been considered before with these general boundary conditions, it has been answered when they consist of interpolation conditions to scattered derivatives (Hermite-Birkhoff interpolation) in Polya [8] and Ferguson [1]. The appropriate condition, the Polya condition, is easily generalized and requires in essence that the total number of conditions on the polynomial and its derivatives up to order $m-1$ shall not be less than m, for all m.

LEMMA 6. *The system of equations* (10) *has a unique polynomial solution, the trivial one, if and only if*

$$n+k+1-r+\operatorname{rank}\|\tilde{A}_{r,m},\ B_{r,m}\|\geqq m,\ \ 1\leqq m\leqq n+1-k. \tag{11}$$

This is always the case if $k\geqq\min(n,\ r-p,\ r-q)$.

PROOF. Abbreviate $\operatorname{rank}\|\tilde{A}_{r,m},\ B_{r,m}\|$ to $\rho(m)$. If for some m, $n+k+1-r+\rho(m)\leqq m-1$ then the system of equations

$$t(x_i)=0, \qquad i=1,\cdots,n+k+1-r$$

$$\sum_{j=0}^{m-1}[a_{ij}t^{(j)}(a)+b_{ij}t^{(n-j)}(b)]=0, \qquad i=1,\cdots,r$$

has rank at most $m-1$. It has therefore a non-trivial solution for $t(x)$ a polynomial of degree at most $m-1$. Clearly this solution also solves (10).

Conversely, let condition (11) hold and assume contrary to the lemma that $t(x)$ is a polynomial of degree exactly $m-1$. Then $\rho(m)=2m$ is impossible since it implies $t(x)\equiv 0$. Thus $2m>\rho(m)\geqq m-(n+k+1-r)$. Using Lemma 3, we then obtain

$$S^+((-1)^it^{(i)}(a))_0^{m-1}+S^+(t^{(i)}(b))_0^{m-1}+S^+(\epsilon t^{(m-1)}(a),\ t^{(m-1)}(b))\geqq\rho(m)$$

where $\epsilon=(-1)^{n+k-m+1+r-\rho(m)}$. Moreover (11) implies

$$\rho(m)\geqq m-(n+k+1-r)+S^+(\epsilon t^{(m-1)}(a),\ t^{(m-1)}(b))$$

because either $\rho(m)=m-(n+k+1-r)$ in which case the last term is zero ($t^{(m-1)}(a)=t^{(m-1)}(b)$), or $\rho(m)\geqq m+1-(n+k+1-r)$ and then it is at most one. Applying Lemma 1, a contradiction with (9) ensues:

$$Z(t;(a,b))\leqq m-1-S^+((-1)^it^{(i)}(a))_0^{m-1}-S^+(t^{(i)}(b))_0^{m-1}\leqq n+k-r.$$

This establishes condition (11) and it remains only to check that it is always satisfied for $k \geqq \min(n, r-p, r-q)$. For $m = n-k+1$, certainly $n+k+1-r+\rho(m) \geqq m$ since $\rho(n+1-k) \geqq r-2k$. Hence, if $k \geqq n$, condition (11) needs no further checking. Next we observe that for any m $\rho(m) \geqq \max(p,q)-(n+1-m)$, since $\operatorname{rank}\|\tilde{A}_{r,m}\| \geqq p-(n+1-m)$ and $\operatorname{rank}\|B_{r,m}\| \geqq q-(n+1-m)$. Thus if $k \geqq \min(r-p, r-q)$ then

$$n+k+1-r+\rho(m) \geqq m+k-\min(r-p, r-q) \geqq m.$$

The proof of Lemma 5 and of the interpolation theorem as well is hereby completed.

4. Examples

It is worth exhibiting some special classes of boundary conditions important in themselves as well as for the purpose of elucidating the present results and putting them in the perspective of the previous literature.

1) In a variety of situations the conditions at the end points are not mixed, it being required that

$$\sum_{j=0}^{n} a_{ij} S^{(j)}(a) = u_i, \qquad i = 1, \cdots, p,$$

$$\sum_{j=0}^{n} b_{ij} S^{(j)}(b) = v_i, \qquad i = 1, \cdots, q.$$

This is the situation considered by Karlin [5] and we recover his results. We have that $p+q=r$ and Postulate I requires $\tilde{A}_{p,n+1}$ to be SC_p and of rank p, $B_{q,n+1}$ to be SC_q and of rank q. The interlacing condition, Formulation 1, now becomes $x_{i-p} < \xi_i < x_{i+n+1-p}$ with the usual exceptions. Moreover the exception of equality at ξ_i can occur only if $N(a,\xi_i] = n+1-p+K(a,\xi_i)$ and $N[\xi_i, b) = n+k+1-q+K(\xi_i, b)$, by the remark following Formulation 2. But this means that the interpolation problem is reducible to one in (a,ξ_i) and one in (ξ_i, b) separately. Practically speaking, it may therefore be assumed that the interlacing condition holds with strict inequality, and that at most $n+1$ data points and knots coincide at any one point.

One final remark concerns the Polya condition (11). In Karlin [5] the condition is stated as follows: there must be indices $i_1 < \cdots < i_p$ and $j_1 < \cdots < j_q$ such that

$$\tilde{A}\binom{1, \cdots, p}{i_1, \cdots, i_p} \neq 0 \quad \text{and} \quad B\binom{1, \cdots, q}{j_1, \cdots, j_q} \neq 0$$

and $j_\sigma \leqq i'_{\sigma+n+k+1-r}, \sigma = 1, \cdots, q-k$, with $\{i'_\sigma\}_1^{n-1\ p}$ the set of indices complementary to the set $\{i_\sigma\}_1^p$ with respect to $\{1, \cdots, n+1\}$. Here $\tilde{A}\binom{1, \cdots, p}{i_1, \cdots, i_p}$ denotes the minor of $\tilde{A}_{p,n+1}$ composed of rows $1, \cdots, p$ and columns $i_1, \cdots, i_p$. The interesting thing about this formulation is that it indicates that the general separated boundary problem (for polynomials) is poised if and only if it is possible to draw from it simple Hermite-Birkhoff interpolation conditions of the kind $S^{(i_\sigma)}(a) = u_{i_\sigma}$, $S^{(j_\tau)}(b) = v_{j_\tau}$, $\sigma = 1, \cdots, p$, $\tau = 1, \cdots, q$, by choosing columns, which yield a poised system. The question arises therefore whether the same is true for the mixed boundary conditions. We want to show that this is indeed the case.

Denote $\rho(l, m) = \operatorname{rank}\|\tilde{A}_{r,l}, B_{r,m}\|$, and let $\tilde{A}_j, B_j$ be the columns of $\tilde{A}_{r,n+1}, B_{r,n+1}$. Consider the matrices $\tilde{A}(m), B(m)$ built as follows: $\tilde{A}(0) = B(0) = 0$ and for $m \geqq 0$

$$\tilde{A}(m+1) = \begin{cases} \|\tilde{A}(m), 0\| & \text{if} \quad \rho(m+1, m) = \rho(m, m) \\ \|\tilde{A}(m), \tilde{A}_{m+1}\| & \text{if} \quad \rho(m+1, m) = \rho(m, m) + 1 \end{cases}$$

$$B(m+1) = \begin{cases} \|0, B(m)\| & \text{if} \quad \rho(m+1, m+1) = \rho(m+1, m) \\ \|B_{n+1-m}, B(m)\| & \text{if} \quad \rho(m+1, m+1) = \rho(m+1, m) + 1. \end{cases}$$

Then $\operatorname{rank}\|\tilde{A}(m), B(m)\| = \rho(m, m)$ and $\|\tilde{A}(n+1), B(n+1)\|$ is a square $r \times r$ matrix when zero columns are discarded. Thus $\|\tilde{A}(n+1), B(n+1)\|$ induces simple Hermite-Birkhoff boundary conditions and Condition (11), reading now $n + k + 1 - r + \operatorname{rank}\|\tilde{A}(m), B(m)\| \geqq m$, ensures its poisedness.

2) Another situation of wide interest is the case of periodic boundary conditions, $S^{(i)}(a) = S^{(i)}(b)$, $i = 0, \cdots, n$. Thus $p = q = r = n + 1$ and

$$\|\tilde{A}_{n+1,n+1}, B_{n+1,n+1}\| = \begin{Vmatrix} (-1)^{n+k} & 0 & \cdot & \cdot & 0 & 0 & \cdot & \cdot & 0 & -1 \\ 0 & (-1)^{n+k-1} & \cdot & \cdot & 0 & 0 & \cdot & \cdot & -1 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & (-1)^k & -1 & \cdot & \cdot & 0 & 0 \end{Vmatrix}$$

It is easily seen that this matrix is SC_{n+1} if and only if k is odd. We consider therefore only periodic splines with an odd number of knots (counting multiplicities).

Since Condition (11) is always satisfied ($r - p = 0$), the interlacing condition is here both necessary and sufficient for the existence of a unique solution. It is illuminating to view the interpolation problem as posed on a circle (i.e. glue the end points together). Formulation 2 of the interlacing condition then requires that on any closed arc I of the circle the number of points in $I, N[I]$, and the number of knots interior to $I, K(I)$, shall satisfy $N[I] \leqq n+1+K(I)$. A condition to this effect was also used by Karlin and Lee [4] in their treatment of generalized periodic splines (which however did not include polynomial splines)

Alternatively we may use Formulation 1, $x_{i-\lambda} < \xi_i < x_{i+n+1-\lambda}$ for some $0 \leqq \lambda \leqq n+1$. As in Example 1, the exception of equality, i.e. the coincidence of more than $n+1$ data points and knots, occurs only in the case of a reducible interpolation problem. For, when this happens say at a knot ξ_i, we may consider the interpolation problem on $(\xi_i, \xi_i + b - a)$, the spline being periodic, and in this interval the spline will satisfy simple Hermite boundary conditions.

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